MATH 1010E University Mathematics
Lecture Notes (week 4)
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1 Derivatives of Piecewise Defined Functions

For piecewise defined functions, we often have to be very careful in computing the derivatives. The differentiation rules (product, quotient, chain rules) can only be applied if the function is defined by ONE formula in a neighborhood of the point where we evaluate the derivative. If we want to calculate the derivative at a point where two different formulas "meet", then we must use the definition of derivative as limit of difference quotient to correctly evaluate the derivative. Let us illustrate this by the following example.

Example 1.1 *Find the derivative $f'(x)$ at every $x \in \mathbb{R}$ for the piecewise defined function*

$$f(x) = \begin{cases} 5 - 2x & \text{when } x < 0, \\ x^2 - 2x + 5 & \text{when } x \geq 0. \end{cases}$$

Solution: We separate into 3 cases: $x < 0$, $x > 0$ and $x = 0$. For the first two cases, the function $f(x)$ is defined by a single formula, so we could just apply differentiation rules to differentiate the function.

$$f'(x) = (5 - 2x)' = -2 \qquad \text{for } x < 0,$$

$$f'(x) = (x^2 - 2x + 5)' = 2x - 2 \qquad \text{for } x > 0.$$

At $x = 0$, we have to use the definition of derivative as limit of difference quotient. First of all,

$$f(0) = 0^2 - 2(0) + 5 = 5.$$

Then we calculate the left-hand and right-hand limits:

$$\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{(5 - 2h) - 5}{h} = \lim_{h \to 0^-} -2 = -2,$$

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{(h^2 - 2h + 5) - 5}{h} = \lim_{h \to 0^+} (h - 2) = -2.$$

Since both of them exists and are equal, we have

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = -2.$$

Therefore, putting all of these together, we see that f is differentiable for every $x \in \mathbb{R}$ and

$$f'(x) = \begin{cases} -2 & \text{when } x \leq 0, \\ 2x - 2 & \text{when } x > 0. \end{cases}$$

Remark 1.2 *From the example above, we see that the derivative $f'(x)$ is still a continuous function (check this!). This is not always true for any function! (Have you seen a counterexample? See Homework 2)*

Example 1.3 *Consider the function defined by*

$$f(x) = \begin{cases} ax + b & \textit{when } x \leq -1, \\ ax^3 + x + 2b & \textit{when } x > -1, \end{cases}$$

for what value(s) of $a, b \in \mathbb{R}$ is the function f differentiable at every $x \in \mathbb{R}$?

Solution: First, it is easy to see that for ANY $a, b \in \mathbb{R}$, the function f is differentiable at every $x \neq -1$ since f is defined by a polynomial on $(-1, +\infty)$ and $(-\infty, -1)$. The only catch is at the point $x = -1$.

If f is differentiable at $x = -1$, it must also be continuous at $x = -1$. Therefore, we need

$$\lim_{x\to -1} f(x) = f(-1).$$

Now, $f(-1) = -a + b$ and the left-hand and right-hand limits are

$$\lim_{x\to -1^-} f(x) = a(-1)^3 + (-1) + 2b = -a + 2b - 1,$$

$$\lim_{x\to -1^+} f(x) = a(-1) + b = -a + b.$$

If f is continuous, then both of these limits must be the same and equal to $f(-1)$. Hence, we have

$$-a + b = -a + 2b - 1 \qquad \Rightarrow \qquad b = 1.$$

Now, we take $b = 1$. To find the value of a which make f differentiable at $x = -1$, we require the limit

$$\lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h}$$

to exists, which is equivalent to the statement that the left-hand and right-hand limits exist and are equal. The left hand limit is

$$\lim_{h\to 0^-} \frac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0^-} \frac{[a(-1+h)+1]-(-a+1)}{h} = \lim_{h\to 0^-} \frac{ah}{h} = a.$$

The right hand limit is

$$\begin{aligned}
\lim_{h\to 0^+} \frac{f(-1+h)-f(-1)}{h} &= \lim_{h\to 0^+} \frac{[a(-1+h)^3+(-1+h)+2]-(-a+1)}{h} \\
&= \lim_{h\to 0^+} \frac{a[(-1+h)^3+1]+h}{h} \\
&= \lim_{h\to 0^+} \frac{ah[(-1+h)^2-(-1+h)+1]+h}{h} \\
&= 3a+1.
\end{aligned}$$

Therefore, if we set them equal to each other, we obtain the condition

$$a = 3a+1 \qquad \Rightarrow \qquad a = -\frac{1}{2}.$$

In summary, we have $a = -1/2$ and $b = 1$ if f is differentiable at every $x \in \mathbb{R}$.

2 Differentiation Rules II: Product and Quotient Rules

Theorem 2.1 *If f and g are differentiable functions, then both their product fg and quotient f/g are differentiable and we have*

(1) Product Rule:

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x),$$

(2) Quotient Rule:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2},$$

provided that $g(x) \neq 0$.

Remark 2.2 *Observe that the differentiation rule $[kf(x)]' = kf'(x)$ where k is a constant is just a special case of product rule by taking $g(x) \equiv k$, which has $g'(x) = 0$.*

Before we go into the proof of these rules, let us first look at a few examples of how to apply these rules to help us calculate derivatives.

Example 2.3

$$\begin{aligned}\frac{d}{dx}[(x+2)(x^2+1)] &= (x+2)'(x^2+1)+(x+2)(x^2+1)' \\ &= (1)(x^2+1)+(x+2)(2x) \\ &= 3x^2+4x+1.\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}[\sin x\cos x] &= (\sin x)'\cos x+\sin x(\cos x)' \\ &= \cos x\cos x+\sin x(-\sin x) \\ &= \cos 2x.\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}\left(\frac{x^2+1}{x+1}\right) &= \frac{(x+1)(x^2+1)'-(x^2+1)(x+1)'}{(x+1)^2} \\ &= \frac{(x+1)(2x)-(x^2+1)(1)}{(x+1)^2} \\ &= \frac{x^2+2x-1}{(x+1)^2}.\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}\left(\frac{\sin x}{x}\right) &= \frac{x(\sin x)'-\sin x(x)'}{x^2} \\ &= \frac{x\cos x-\sin x}{x^2}.\end{aligned}$$

Note that for the last two examples, the calculation is valid only for $x \neq -1$ and $x \neq 0$ respectively.

Question: Does the limit $\lim_{x\to 0}\frac{d}{dx}\left(\frac{\sin x}{x}\right)$ exist? If we define

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{when } x \neq 0, \\ 1 & \text{when } x = 0, \end{cases}$$

then does $f'(0)$ exists? If so, is it related to the limit at the beginning?

Question: Calculation the derivatives of all the trigonometric and hyperbolic functions!

Now, we come back to the proof of the product rule and quotient rule.

Proof of Product Rule: Using the definition of derivative,

$$\begin{aligned}
[f(x)g(x)]' &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h)g(x+h) - f(x)g(x+h)] + [f(x)g(x+h) - f(x)g(x)]}{h} \\
&= \lim_{h\to 0} \left(g(x+h)\ \frac{f(x+h) - f(x)}{h} \right) + \lim_{h\to 0} \left(f(x)\ \frac{g(x+h) - g(x)}{h} \right) \\
&= g(x)f'(x) + f(x)g'(x).
\end{aligned}$$

In the last equality, we have also used that $\lim_{h\to 0} g(x+h) = g(x)$ since g is continuous (even differentiable) by assumption.

Proof of Quotient Rule: Using the definition of derivative,

$$\begin{aligned}
\left(\frac{f(x)}{g(x)}\right)' &= \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h} - \frac{f(x)}{g(x)}}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x)g(x+h)} \\
&= \lim_{h\to 0} \frac{\frac{f(x+h)g(x) - f(x)g(x)}{h} + \frac{f(x)g(x) - f(x)g(x+h)}{h}}{g(x)g(x+h)} \\
&= \lim_{h\to 0} \frac{g(x)\ \frac{f(x+h)-f(x)}{h} - f(x)\ \frac{g(x+h)-g(x)}{h}}{g(x)g(x+h)} \\
&= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.
\end{aligned}$$

Again we have used the continuity of g in the last equality.

3 Composite Functions

Apart from addition, subtraction, multiplication and division to get new functions, there is another useful way to obtain new functions from old called *composition*.

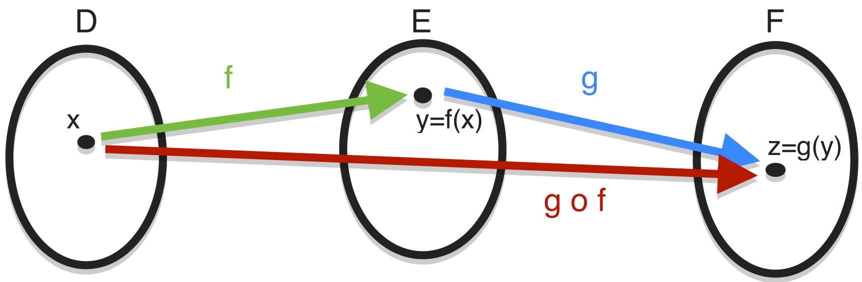
Definition 3.1 *Given two functions $f : D \to E$ and $g : E \to F$, we can define the* composite function

$$g \circ f : D \to F \qquad \textit{by} \qquad g \circ f(x) := g(f(x)).$$

If we think of functions as assignments, then $g \circ f$ means assigning x to $f(x)$ first and then we further assign $f(x)$ to $g(f(x))$. Note that in order for the composition $g \circ f$ to be well defined, the domain of g must contain the image of f. However, we do not require either f or g to be injective or surjective.

Question: Which of the following statement is true?

- f and g are injective $\Rightarrow g \circ f$ is injective?
- f and g are injective $\Rightarrow g \circ f$ is injective?
- g is not injective $\Rightarrow g \circ f$ is not injective?
- g is not surjective $\Rightarrow g \circ f$ is not surjective?



Let us look at one example. Consider two functions defined by

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) := \cos x,$$

$$g : \mathbb{R} \to \mathbb{R}, \qquad g(y) := y^2,$$

the composition $g \circ f : \mathbb{R} \to \mathbb{R}$ is hence

$$g \circ f(x) = g(f(x)) = g(\cos x) = \cos^2 x.$$

Note that we can also do the composition in a different order for this example. We can form $f \circ g : \mathbb{R} \to \mathbb{R}$ which is defined by

$$f \circ g(y) = f(g(y)) = f(y^2) = \cos y^2.$$

Note that even in this case both $g \circ f$ and $f \circ g$ are defined, they are NOT equal to each other. Therefore, the ordering is important when we talk about composition:

Remark 3.2 *In general, we have $g \circ f \neq f \circ g$ even when both are well-defined.*

The notation $g \circ f$ may be confusing sometimes since we are writing g first and then f but the composition means doing f first and then g. One good way to memorize this is that when $g \circ f$ acts on x, we write $g \circ f(x)$. It is f which hits x first, and then followed by g. This is the reason why we use this convention.

Also, in the above example, we write $g \circ f$ as a function of x and $f \circ g$ as a function of y. When comparing them as functions, the name of the variable (x and y) are irrelevant. It is the rule of assignment that determines the function. For example, we consider $f(x) = x$ and $g(y) = y$ as the same "function". It will become clear later that it is useful to keep using x for elements in the domain of f and y as elements in the codomain of f.

Composition is a rather nice operation which preserves many of the analytic properties of f and g.

Theorem 3.3 *If*

(i) f is continuous at x_0, and

(ii) g is continuous at $y_0 := f(x_0)$,

then $g \circ f$ is continuous at x_0. In other words, composition of continuous functions is continuous.

The "proof" is rather intuitive. When x is close to x_0, then $y := f(x)$ is also close to $y_0 := f(x_0)$ by the continuity of f. On the other hand, since y is close to y_0, we must have $g(y)$ close to $g(y_0)$. This is the same as saying that $g(f(x))$ is close to $g(f(x_0))$. A rigorous proof can given using the $\epsilon - \delta$ definition of continuity. Interested students can try to write down a complete proof as an exercise.

We have already used this theorem implicitly many times when we evaluate limits. For example, when we compute

$$\lim_{x \to 0} \cos^2 x = (\cos 0)^2 = 1,$$

we have used the continuity of cos and square function which justifies the direct substitution with $x = 0$ to compute the limit.

4 Differentiation Rules III: Chain Rule

Composition of differentiable functions is also differentiable.

Theorem 4.1 (Chain Rule) *If*

(i) f *is differentiable at* x_0*, and*

(ii) g *is differentiable at* $y_0 := f(x_0)$*,*

then $g \circ f$ *is differentiable at* x_0 *and*

$$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0).$$

Remark 4.2 *The chain rule simply says that the derivative of composition is just the product of derivatives. Yet this is one of the most confusing rules among all the differentiation rules we have seen so far. The reason is that we have to be careful at which points we are evaluating the derivatives! We are NOT evaluating* g' *at* x_0*, but at* $f(x_0)$ *instead. If you recall the step-by-step assignment picture of composite functions, you see that it is indeed the only way for the formula to make sense since* $g'(x_0)$ *is not even defined as* x_0 *is not in the domain of* g*!*

Let us turn to some examples.

Example 4.3 *Calculate the derivatives of the following functions:*

(i) $\cos x^2$,

(ii) $(x^2+1)^7$,

(iii) $e^{\sin x}$,

(iv) $e^{\sin x^2}$.

Solution: (i) Let $f(x) = x^2$ and $g(y) = \cos y$, then $g \circ f(x) = \cos x^2$ is the function we want to differentiate. Note that

$$f'(x) = 2x \qquad \text{and} \qquad g'(y) = -\sin y.$$

Therefore, applying chain rule we get

$$(g \circ f)'(x) = g'(f(x))f'(x) = (-\sin(x^2)) \cdot (2x) = -2x \sin x^2.$$

(ii) Let $f(x) = x^2+1$ and $g(y) = y^7$. We have $f'(x) = 2x$ and $g'(y) = 7y^6$, Therefore,

$$\frac{d}{dx}(x^2+1)^7 = 7(x^2+1)^6 \cdot (2x) = 14x(x^2+1)^6.$$

Therefore, the chain rule basically says that we can differentiate "layer-by-layer", starting from the outermost layer. In this example, we first differentiation the function of "raising to power 7" and then differentiate into the bracket.

(iii) Using the concept of differentiating layer-by-layer, we do not need to define f and g every time. Since we know the derivative of e^y is just e^y and the derivative of $\sin x$ is $\cos x$, we get

$$\frac{d}{dx}e^{\sin x} = e^{\sin x} \cdot \cos x.$$

(iv) For functions involving more than two layers, we just differentiate them one by one. Here, the outer layer is exponential function, the middle layer is sin and the inner later is x^2, so

$$\frac{d}{dx}e^{\sin x^2} = e^{\sin x^2} \cdot (\cos x^2) \cdot (2x).$$

Note that when differentiating the outer layers, you keep the inner layers unchanged. This makes sure that you are evaluating at the correct point as discussed in Remark 4.2.

Question: Use layer by layer differentiation to evaluate the following derivatives:

$$\frac{d}{dx}\sqrt{x+\sqrt{x}} \qquad \text{and} \qquad \frac{d}{dx}\left(\frac{x}{\sqrt{1+x^2}}\right).$$

Proof of Chain Rule: By definition of derivative,

$$\begin{aligned}(g \circ f)'(x_0) &= \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} \\ &= \lim_{x \to x_0} \left[\frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \frac{f(x) - f(x_0)}{x - x_0}\right] \\ &= \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.\end{aligned}$$

Let $y = f(x)$ and $y = f(x_0)$, since f is continuous at x_0, we have $y \to y_0$ as $x \to x_0$, therefore, we can rewrite the first limit in the last line above in terms of y:

$$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} \cdot \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = g'(y_0)f'(x_0) = g'(f(x_0))f'(x_0).$$

This proves the chain rule.

Question: There is a loophole in the above proof. Can you find it and fix it?

5 Inverse Differentiation

We can use the chain rule to find the derivative of the inverse f^{-1} of a function, if it exists.

Theorem 5.1 *Let $f : (a, b) \to (c, d)$ be a bijective differentiable function whose derivative f' is continuous. Then, the inverse function $g : (c, d) \to (a, b)$ is differentiable at every y such that $f'(g(y)) \neq 0$ and*

$$g'(y) = \frac{1}{f'(g(y))}.$$

Remark 5.2 *Note that the inverse may not be differentiable at where $f' = 0$. For example, consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$, then its inverse $g(y) = y^{1/3}$ exists but is not differentiable at $y = 0$.*

Proof of Theorem 5.1 The proof that g is differentiable is much more involved so we skip it here. By the definition of inverse,

$$f(g(y)) = y \qquad \text{for all } y.$$

Since the above equation holds for ALL y, we can differentiate the whole equation on both side with respect to y, applying chain rule on the left hand side, we obtain

$$f'(g(y))g'(y) = 1.$$

If $f'(g(y)) \neq 0$, we can divide it to the other side to obtain the formula of $g'(y)$.

Example 5.3 *Show that $\frac{d}{dx}(ln\ y) = \frac{1}{y}$.*

Solution: Recall that the exponential function $\exp(x) : \mathbb{R} \to (0, \infty)$ is 1-1 and onto whose inverse is $\ln y : (0, \infty) \to \mathbb{R}$. We have already seen that $\frac{d}{dx}\exp(x) = \exp(x)$. Therefore, using inverse differentiation, keeping in mind that $y = \exp(x)$, we have

$$\frac{d}{dy}\ln(y) = \frac{1}{\exp(x)} = \frac{1}{y}.$$

Example 5.4 *Show that $\frac{d}{dx}\sin^{-1} y = \frac{1}{\sqrt{1-y^2}}$.*

Solution: Recall that $\sin(x) : \mathbb{R} \to \mathbb{R}$ is neither 1-1 nor onto so the inverse does not exists. However, if we restrict its domain and codomain to $\sin(x) : (-\frac{\pi}{2}, \frac{\pi}{2}) \to (-1, 1)$, then it becomes 1-1 and onto and hence there is an inverse $\sin^{-1} y : (-1, 1) \to (-\frac{\pi}{2}, \frac{\pi}{2})$. Using inverse differentiation, since $y = \sin x$, we have

$$\frac{d}{dy} \sin^{-1} y = \frac{1}{\frac{d}{dx} \sin x} = \frac{1}{\cos x} = \frac{1}{\sqrt{1 - \sin^2 x}} = \frac{1}{\sqrt{1 - y^2}}.$$

Note that we could have restrict $\sin(x)$ to a different domain e.g. $(\frac{\pi}{2}, \frac{3\pi}{2})$. The inverse function $\sin^{-1} y$ would be different, but the derivative is the same! The choice of $\sin^{-1}$ is a phenomenon called "branching".

Question: What happens to $\frac{d}{dy} \sin^{-1} y$ when $y \to \pm 1$? What does it correspond to in terms of the slope of the graphs?

Question: Discuss the domains and codomains of the inverses of other trigonometric and hyperbolic functions. What are their derivatives?

6 More Examples

We give a few more examples illustrating the use of all the differentiation rules we discussed.

$$\begin{aligned} \frac{d}{dx}(\ln(x + \sqrt{1 + x^2})) &= \frac{1}{x + \sqrt{1 + x^2}} \frac{d}{dx}(x + \sqrt{1 + x^2}) \\ &= \frac{1}{x + \sqrt{1 + x^2}} [1 + x(1 + x^2)^{-\frac{1}{2}}] \\ &= \frac{1}{\sqrt{1 + x^2}}. \end{aligned}$$

We can use exp and ln to define an arbitrary exponential function a^b for ANY $a > 0$ and $b \in \mathbb{R}$:

$$a^b := \exp(b \ln a).$$

Hence, we can calculate the derivatives of these general exponential functions:

$$\frac{d}{dx} 3^x = \frac{d}{dx}(\exp(x \ln 3)) = \exp(x \ln 3) \cdot \ln 3 = 3^x \ln 3.$$

$$\frac{d}{dx} x^x = \frac{d}{dx}(\exp(x \ln x)) = \exp(x \ln x) \cdot [(1)(\ln x) + (x)(\frac{1}{x})] = (1 + \ln x) x^x.$$

Question: Calculate $\frac{d}{dx}(x^{x^x})$.